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# Deformed Minkowski spaces: classification and properties

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**Abstract.** Using general but simple covariance arguments, we classify the 'quantum' Minkowski spaces for dimensionless deformation parameters. This requires a previous analysis of the associated Lorentz groups, which reproduces a previous classification by Woronowicz and Zakrzewski. As a consequence of the unified analysis presented, we give the commutation properties, the deformed (and central) length element and the metric tensor for the different spacetime algebras.

## 1. Introduction

Following the approach of [1], we present here a classification of the possible deformed Minkowski spaces (algebras). Our analysis, which provides a common framework for the properties of the various Minkowski spacetimes, requires the consideration of the two ($SL_q(2)$ and $SL_h(2)$) deformations of $SL(2, C)$ and provides a characterization of the appropriate $R$-matrices defining the deformed Lorentz groups given in [2] (see also [3]).

It is well known that $GL(2, C)$ admits only two different deformations that possess a central determinant: one is the standard $q$-deformation [4, 5] and the other is the non-standard or 'Jordanian' $h$-deformation [6–8]. Both $GL_q(2)$ and $GL_h(2)$ have associated 'quantum spaces' in the sense of [9]. These deformations (which may be shown to be related by contraction [10]) are defined as the associative algebras generated by the entries $a, b, c, d$ of a matrix $M$, the commutation properties of which may be expressed by an 'FRT' equation [5]

$$R_{12}M_1M_2 = M_2M_1R_{12} \tag{1}$$

for a suitable $R$-matrix. Let us summarize their properties.

*(a)* For $GL_q(2)$ the $R$-matrix in (1) is ($\lambda \equiv q - q^{-1}$)

$$R_q = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \qquad \hat{R}_q \equiv \mathcal{P}R_q = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \qquad \mathcal{P}R_q\mathcal{P} = R_q^t \tag{2}$$

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where $\mathcal{P}$ is the permutation operator ($\mathcal{P} = \mathcal{P}^\dagger$, $\mathcal{P}_{ij,kl} = \delta_{il}\delta_{jk}$), and the commutation relations defining the quantum group algebra are

$$\begin{array}{lll} ab = qba & ac = qca & ad - da = \lambda bc \\ bc = cb & bd = qdb & cd = qdc. \end{array} \tag{3}$$

$Fun(GL_q(2))$ has a quadratic central element,

$$\det_q M := ad - qbc \tag{4}$$

where $\det_q M = 1$ defines $SL_q(2)$. The matrix $\hat{R}_q \equiv \mathcal{P}R_q$ satisfies Hecke's condition

$$\hat{R}_q^2 - \lambda\hat{R}_q - I = 0 \qquad (\hat{R}_q - qI)(\hat{R}_q + q^{-1}I) = 0 \tag{5}$$

and (we shall assume $q^2 \neq -1$ [5] throughout) it has a spectral decomposition in terms of a rank-three projector $P_{q+}$ and a rank-one projector $P_{q-}$,

$$\begin{array}{ll} \hat{R}_q = qP_{q+} - q^{-1}P_{q-} & \hat{R}_q^{-1} = q^{-1}P_{q+} - qP_{q-} \\ [\hat{R}_q, P_{q\pm}] = 0 & P_{q\pm}\hat{R}_q P_{q\mp} = 0 \end{array} \tag{6}$$

$$P_{q+} = \frac{I + q\hat{R}_q}{1+q^2} \qquad P_{q-} = \frac{I - q^{-1}\hat{R}_q}{1+q^{-2}} = \frac{1}{1+q^{-2}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-2} & -q^{-1} & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

The following relations have an obvious equivalent in the undeformed case:

$$\epsilon_q M^t \epsilon_q^{-1} = M^{-1} \qquad \epsilon_q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} = -\epsilon_q^{-1} \qquad P_{q-\,ij,kl} = \frac{-1}{[2]_q}\epsilon_{q\,ij}\epsilon^{-1}_{q\,kl}\,. \tag{8}$$

The determinant of an ordinary $2\times 2$ matrix may be defined as the proportionality coefficient in $(\det M)P_- := P_- M_1 M_2$ where $P_-$ is given by (7) for $q$=1. In the $q \neq 1$ case the $q$-determinant (4) may be expressed as

$$(\det_q M)P_{q-} := P_{q-}M_1M_2 \qquad (\det_q M^{-1})P_{q-} = M_2^{-1}M_1^{-1}P_{q-} \tag{9}$$

($\det_q M^{-1} = (\det_q M)^{-1}$ and $(\det_q M)^\dagger P_{q-}^\dagger = M_2^\dagger M_1^\dagger P_{q-}^\dagger$).

*(b)* For $GL_h(2)$ the $R$-matrix in (1) is the solution of the Yang–Baxter equation given by

$$R_h = \begin{bmatrix} 1 & -h & h & h^2 \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \hat{R}_h \equiv \mathcal{P}R_h = \begin{bmatrix} 1 & -h & h & h^2 \\ 0 & 0 & 1 & h \\ 0 & 1 & 0 & -h \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathcal{P}R_h\mathcal{P} = R_h^{-1} \tag{10}$$

(or $R_{h\,12}R_{h\,21} = I$, the triangularity condition) for which (1) gives

$$\begin{array}{lll} [a,b] = h(\xi - a^2) & [a,c] = hc^2 & [a,d] = hc(d-a) \\ [b,c] = h(ac+cd) & [b,d] = h(d^2 - \xi) & [c,d] = -hc^2 \end{array} \tag{11}$$

(so that $[a-d, c] = 0$ follows), where $\xi$ is the quadratic central element

$$\xi \equiv \det_h M = ad - cb - hcd \,. \tag{12}$$

Setting $\xi = 1$ reduces $GL_h(2)$ to $SL_h(2)$. The matrix $\hat{R}_h$ satisfies

$$\hat{R}_h^2 = I \qquad (I - \hat{R}_h)(I + \hat{R}_h) = 0 \,. \tag{13}$$

It has two eigenvalues (1 and $-1$) and a spectral decomposition in terms of a rank-three projector $P_{h+}$ and a rank-one projector $P_{h-}$

$$\hat{R}_h = P_{h+} - P_{h-} \qquad P_{h\pm}\hat{R}_h = \pm P_{h\pm} \tag{14}$$

$$P_{h+} = \frac{1}{2}(I + \hat{R}_h) \qquad P_{h-} = \frac{1}{2}(I - \hat{R}_h) = \frac{1}{2}\begin{bmatrix} 0 & h & -h & -h^2 \\ 0 & 1 & -1 & -h \\ 0 & -1 & 1 & h \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{15}$$

For $SL_h(2)$, the formulae equivalent to those in (8) are

$$\begin{aligned} &\epsilon_h M^t \epsilon_h^{-1} = M^{-1} \qquad \epsilon_h = \begin{pmatrix} h & 1 \\ -1 & 0 \end{pmatrix} \\ &\epsilon_h^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix} \qquad P_{h-\,ij,kl} = \frac{-1}{2}\epsilon_{h\,ij}\epsilon^{-1}_{h\,kl} \,. \end{aligned} \tag{16}$$

Using $P_{h-}$, the deformed determinant and its inverse, $\det_h M$ and $\det_h M^{-1}$, (12) are also given by equations (9).

The quantum planes [9] associated with $SL_q(2)$ and $SL_h(2)$ are the associative algebras generated by two elements $(x, y) \equiv X$, the commutation properties of which (explicitly and in $R$-matrix form) are:

(a) for $SL_q(2)$ [9]

$$xy = qyx \quad \longleftrightarrow \quad R_q X_1 X_2 = q X_2 X_1 \tag{17}$$

(b) for $SL_h(2)$ [7, 8]

$$xy = yx + hy^2 \quad \longleftrightarrow \quad R_h X_1 X_2 = X_2 X_1 \,. \tag{18}$$

These commutation relations are preserved under transformations by the corresponding quantum groups matrices† $M$, $X' = MX$. This invariance statement, suitably extended to apply to the case of deformed Minkowski spaces, provides the essential ingredient for their classification.

From now on we shall often write $R_Q$, $P_Q$ ($Q = q, h$) to treat both deformations simultaneously. For instance, equations (17) and (18) may be jointly written as $R_Q X_1 X_2 = \rho X_2 X_1$, where $\rho = (q, 1)$ is the appropriate eigenvalue of $R_Q$.

† The $GL_q(2)$ and $GL_h(2)$ matrices also preserve the '$q$-symplectic' and '$h$-symplectic' metrics $\epsilon_q$ (or ${\epsilon_q}^{-1}$) and ${\epsilon_h}^{-1}$, respectively.

## 2. Deformed Lorentz groups and associated Minkowski algebras

As is well known, the vector representation $D^{\frac{1}{2},\frac{1}{2}} = D^{\frac{1}{2},0} \otimes D^{0,\frac{1}{2}}$ of the restricted Lorentz group may be given by the transformation $K' = AKA^\dagger$, $A \in SL(2,C)$. The spacetime coordinates are contained in $K = K^\dagger = \sigma^\mu x_\mu$, where $\sigma^0 = I$ and $\sigma^i$ are the Pauli matrices; the time coordinate may be identified as $x^0 = \frac{1}{2}\,\mathrm{tr}(K)$. Since $\det K = (x_0)^2 - x^i x_i = \det K'$, the correspondence $\pm A \mapsto \Lambda \in SO(1,3)$, where $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, realizes the covering homomorphism $SL(2,C)/Z_2 = SO(1,3)$. A first step to obtain a deformation of the Lorentz group is to replace the $SL(2,C)$ matrices $A$ above by the generator matrix $M$ of $SL_q(2)$ [11–14].

In general, the full determination of a deformed Lorentz group requires the characterization of all possible commutation relations among the generators $(a,b,c,d)$ of $M$ and $(a^*,b^*,c^*,d^*)$ of $M^\dagger$, $M$ being a deformation of $SL(2,C)$. The $R$-matrix form of these may be expressed in full generality by

$$\begin{aligned} &R^{(1)}M_1M_2 = M_2M_1R^{(1)} \qquad &&M_1^\dagger R^{(2)}M_2 = M_2R^{(2)}M_1^\dagger \\ &M_2^\dagger R^{(3)}M_1 = M_1R^{(3)}M_2^\dagger \qquad &&R^{(4)}M_1^\dagger M_2^\dagger = M_2^\dagger M_1^\dagger R^{(4)} \end{aligned} \tag{19}$$

where $R^{(3)\,\dagger} = R^{(2)} = \mathcal{P}R^{(3)}\mathcal{P}$ (or the 'reality' condition† for $R^{(3)}$) and $R^{(4)} = R^{(1)\,\dagger}$ or $R^{(4)} = (\mathcal{P}R^{(1)\,-1}\mathcal{P})^\dagger$ since the first equation in (19) is invariant under the exchange $R^{(1)} \leftrightarrow \mathcal{P}R^{(1)\,-1}\mathcal{P}$.

Equations (19), which also follow (see, e.g., [15]) from the bi-spinor (dotted and undotted) description of 'quantum' spacetime in terms of a deformed $K$, will be taken as the starting point for the classification of the deformed Lorentz groups. In it, the matrix $R^{(1)}$ characterizes the appropriate deformation of the $SL(2,C)$ group ($R^{(1)} = R_Q$), $R^{(2)}$ (or $R^{(3)}$) defines how the elements of $M$ and $M^\dagger$ commute and it is not *a priori* fixed (but it must satisfy consistency relations with $R^{(1)}$, see equation (20) below) and $R^{(4)}$ gives the commutation relations for the complex conjugated generators contained in $M^\dagger$. The specification of the deformed Lorentz group will be completed by the commutation properties of the generators with their complex conjugated ones, i.e. by the determination of $R^{(2)} = R^{(3)\,\dagger}$.

The commutation relations of the deformed Lorentz group algebra generators (entries of $M$ and $M^\dagger$) are given by equations (19). The consistency of these relations is assured if $R^{(1)}$ (and $R^{(4)}$) obey the Yang–Baxter equation (YBE) and $R^{(3)}$ and $R^{(2)}$ satisfy the mixed consistency equations [1, 16]

$$R_{12}^{(1)}R_{13}^{(3)}R_{23}^{(3)} = R_{23}^{(3)}R_{13}^{(3)}R_{12}^{(1)} \qquad R_{12}^{(4)}R_{13}^{(2)}R_{23}^{(2)} = R_{23}^{(2)}R_{13}^{(2)}R_{12}^{(4)} \tag{20}$$

(these two equations are actually the same since either $R^{(4)} = R^{(1)\,\dagger}$ or $R^{(4)} = (\mathcal{P}R^{(1)\,-1}\mathcal{P})^\dagger$ and $R^{(2)} = R^{(3)\,\dagger}$). It will be convenient to notice that the first equation, considered as an 'RTT' equation, indicates that $R^{(3)}$ is a representation of the deformed $GL(2,C)$ group, i.e. the matrix $R^{(3)}$ provides a $2\times2$ representation of the entries $M_{ij}$ of the generator matrix $M$: $(M_{ij})_{\alpha\beta} = R^{(3)}_{i\alpha,j\beta}$. Thus, $R^{(3)}$ may be seen as a matrix in which the $2\times2$ blocks satisfy between themselves the same commutation relations that the entries of $M$ do

$$R^{(3)} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \sim \quad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

† This reality condition can be given in a more general form $R^{(3)\,\dagger} = \tau\mathcal{P}R^{(3)}\mathcal{P}$ for $|\tau| = 1$; however, this phase factor can be eliminated by the redefinition $R^{(3)} \to \tau^{1/2}R^{(3)}$ (cf [2]).

and the problem of finding all possible Lorentz deformations is equivalent to finding all possible $R^{(3)}$ matrices with $2\times2$ block entries satisfying (3) or (11) such that $\mathcal{P}R^{(3)}\mathcal{P} = R^{(3)\dagger}$ ($\hat{R}^{(3)} = \hat{R}^{(3)\dagger}$).

To introduce the deformed Minkowski *algebra* $\mathcal{M}^{(j)}$ associated with a deformed Lorentz group $L^{(j)}$ (where the index $j$ refers to the different cases) it is natural to extend $K' = AKA^\dagger$ above to the deformed case by stating that in it the corresponding $K$ generates a comodule algebra for the coaction $\phi$ defined by

$$\phi : K \longmapsto K' = MKM^\dagger \qquad K'_{is} = M_{ij}M^\dagger_{ls}K_{jl} \qquad K = K^\dagger \qquad \Lambda = M \otimes M^* \tag{21}$$

where it is assumed that the matrix elements of $K$, which now do not commute among themselves, commute with those of $M$ and $M^\dagger$. As in (17), (18) for $q$-two-vectors (rather, two-*spinors*) we now demand that the commuting properties of the entries of $K$ are preserved by (21). Extensive use has been made of covariance arguments to characterize the algebra generated by the entries of $K$, and the resulting equations are associated with the name of reflection equations [17, 18] or, in a more general setting, braided algebras [19, 20] of which the former constitute the 'algebraic sector' (for an introduction to braided geometry see [21]); similar equations were also early introduced in [16]. Let us now extend the arguments given in [1] to classify the deformed Lorentz groups and their associated Minkowski algebras in an unified way.

This is achieved by describing the commutation properties of the entries of the Hermitian matrix $K$ generating a possible Minkowski algebra $\mathcal{M}$ by means of a general reflection equation of the form

$$R^{(1)}K_1R^{(2)}K_2 = K_2R^{(3)}K_1R^{(4)} \tag{22}$$

where the $R^{(i)}$ matrices ($i = 1, \ldots, 4$) are those introduced in (19). Indeed, writing equation (22) for $K' = MKM^\dagger$, it follows that the invariance of the commutation properties of $K$ under the associated deformed Lorentz transformation (21) is achieved if relations (19) are satisfied.

The deformed Minkowski length and metric, invariant under a Lorentz transformation (21) of $L^{(j)}$, is defined through the quantum determinant of $K$. Since the two matrices $\hat{R}^{(1)} = \mathcal{P}R_Q$ have spectral decompositions (equations (6), (14)) with a rank-three projector $P_{Q+}$ and a rank-one projector $P_{Q-}$, and the determinants of $M$, $M^\dagger$ are central (equations (9), (12)), the *Q-deformed* and *invariant* (under (21)) determinant of the $2\times2$ matrix $K$ may now be given by

$$(\det_Q K)P_{Q-}P^\dagger_{Q-} = -\rho P_{Q-}K_1\hat{R}^{(3)}K_1P^\dagger_{Q-}\,. \tag{23}$$

It is easy to check that $(P_{Q-}P^\dagger_{Q-})^2 = \left(\omega_Q/|[2]_\rho|\right)^2 P_{Q-}P^\dagger_{Q-}$, where $\omega_q = |q| + |q^{-1}|$, $\omega_h = 2 + h^2$ and $[2]_1 = 2$. In (23), the subindex $Q$ in $\det_Q K$ indicates that it depends on $q$ or $h$ (or on other parameters on which $R^{(3)}$ may depend) and $\rho$ ($= (q, 1)$ as before) has been added by convenience. Since $\hat{R}^{(3)}$ and $K$ are Hermitian, $\det_Q K$ is real (if $\rho$ is not real it may be factored out). We stress that the above formula provides a general expression for a central (see below) quadratic element which constitutes the *deformed Minkowski length* for *all* deformed spacetimes $\mathcal{M}^{(j)}$.

Similarly, it is possible to write in general the invariant scalar product of *contravariant* (transforming as the matrix $K$, equation (21)) and *covariant* (transforming by $Y \mapsto Y' = (M^\dagger)^{-1}YM^\dagger$) matrices (four-vectors) as the quantum trace of a matrix product [1] (cf [5]). In the present general case, the deformed trace of a matrix $B$ is defined by

$$\mathrm{tr}_Q(B) := \mathrm{tr}(\mathcal{D}_Q B) \qquad \mathcal{D}_Q = \rho^2\,\mathrm{tr}_{(2)}(\mathcal{P}(((R_Q)^{t_1})^{-1})^{t_1}) \tag{24}$$

where $\mathrm{tr}_{(2)}$ means trace in the second space. This deformed trace is invariant under the quantum group coaction $B \mapsto MBM^{-1}$ since the expression of $\mathcal{D}_Q$ above guarantees that $\mathcal{D}_Q^t = M^t \mathcal{D}_Q^t (M^{-1})^t$ is fulfilled. In particular, the $\mathcal{D}_Q$ matrices for $R_q$ and $R_h$ are found to be

$$\mathcal{D}_q = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \qquad \mathcal{D}_h = \begin{pmatrix} 1 & -2h \\ 0 & 1 \end{pmatrix}. \tag{25}$$

Let us now find the expression of the metric tensor. Using $\epsilon_Q$ (cf equations (8), (16)) $(P_{Q-})_{ij,kl} = -(1/[2]_\rho)\epsilon_{Q\,ij}\epsilon^{-1}_{Q\,kl}$ and $\mathcal{D}_Q = -\epsilon_Q(\epsilon_Q^{-1})^t$ ($\mathcal{D}_Q^t = M^t \mathcal{D}_Q^t (M^{-1})^t$ now follow from $\epsilon_Q M^t \epsilon_Q^{-1} = M^{-1}$, equations (8) and (16)). The covariant $K^\epsilon_{ij}$ vector is

$$K^\epsilon_{ij} = \hat{R}^\epsilon_{Q\,ij,kl} K_{kl} \qquad \hat{R}^\epsilon_Q \equiv (1 \otimes (\epsilon_Q^{-1})^t)\hat{R}^{(3)}(1 \otimes (\epsilon_Q^{-1})^\dagger) \tag{26}$$

from which follows that the general Minkowski length and metric is given by

$$l_Q \equiv \det_Q K = \frac{\rho}{\omega_Q} \mathrm{tr}_Q\, K K^\epsilon \equiv \rho^2 g_{Q\,ij,kl} K_{ij} K_{kl} \qquad g_{Q\,ij,kl} = \frac{\rho^{-1}}{\omega_Q} \mathcal{D}_{Q\,si} \hat{R}^\epsilon_{Q\,js,kl}\,. \tag{27}$$

This concludes the unified description of all cases. Let us now look at their classification and specific properties.

## 3. Characterization of the Lorentz deformations

First we use the reality condition $R^{(3)\dagger} = \mathcal{P}R^{(3)}\mathcal{P}$ to reduce the number of independent parameters in $R^{(3)}$. It implies

$$R^{(3)} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a^*_{21} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ a^*_{12} & c_{12} & a^*_{22} & c^*_{22} \\ b^*_{12} & c_{22} & b^*_{22} & d_{22} \end{bmatrix} \tag{28}$$

where $a_{11}$, $d_{22}$, $b_{21}$, $c_{12}$ are real numbers and the rest are complex.

*(a) Deformed Lorentz groups associated with $SL_q(2)$*

Let now $M \in SL_q(2)$ and $R^{(1)} = R_q$, equation (2). The problem of finding the $q$-Lorentz groups associated with the standard deformation is now reduced to obtaining all matrices $R^{(3)}$ satisfying (20). This means that the $2\times2$ matrices $A, B, C, D$ in (28) must satisfy the commutation relations in (3). This implies that (see [2]) $B^2 = C^2 = 0$, $AD \sim I_2$ and that either $B$ or $C$ are zero. Now

*(a1)* $B = 0$ gives:

$$R^{(3)} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & c_{12} & a^*_{22} & 0 \\ 0 & 0 & 0 & d_{22} \end{bmatrix} \qquad \text{with } a_{11},\ d_{22},\ c_{12} \in R\,.$$

From $AD \sim I_2$ it is easy to see (fixing first $a_{11} = 1$) that $d_{22} = a^*_{22}/a_{22}$; its reality then implies $d_{22} = \pm 1$, $d_{22} = 1$ when $a_{22} \in R$ and $d_{22} = -1$ for $a_{22} \in \mathrm{i}R$. The relation $AC = qCA$ forces $a_{22} = q^{-1}$ or $c_{12} = 0$.

*(a2)* $C = 0$ gives:

$$R^{(3)} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & b_{21} & 0 \\ 0 & 0 & a_{22}^* & 0 \\ 0 & 0 & 0 & d_{22} \end{bmatrix} \qquad \text{with } a_{11},\, d_{22},\, b_{21} \in R,\; a_{11} = 1\,.$$

as in the previous case, $d_{22} = \pm 1$ and $a_{22} \in R$ for $d_{22} = 1$ and $a_{22} \in \mathrm{i}R$ for $d_{22} = -1$. Analogously, from $AB = qBA$ one obtains that $b_{21} = 0$ or $a_{22} = q$.

Thus, the solutions for $R^{(3)}$ are as follows:

$$R^{(3)} = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & r & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \qquad q \in R \quad r \in R \tag{29}$$

$$R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & \pm t & 0 \\ 0 & 0 & 0 & \pm 1 \end{bmatrix} \qquad + \text{ for } t \in R \quad - \text{ for } t \in \mathrm{i}R \tag{30}$$

$$R^{(3)} = \begin{bmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{bmatrix} \qquad q \in R \quad r \in R \tag{31}$$

$$R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & r & -q^{-1} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad q \in \mathrm{i}R \quad r \in R \tag{32}$$

$$R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q & r & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad q \in \mathrm{i}R \quad r \in R \tag{33}$$

*Remarks.* Notice that, as anticipated, the $Q$-'determinant' of all these $R^{(3)}$ matrices, computed as $\det_Q M$, is a scalar (and hence a commuting) $2\times 2$ matrix.

$R_q^\dagger = \mathcal{P} R_q \mathcal{P}$ iff $q \in R$. Hence, $R_{12}^{(4)} = R_{q21}$ or $R_{q12}^{-1}$. Thus $\tilde{M} \equiv (M^{-1})^\dagger$ provides a second copy of $SL_q(2)$, since then $R_q \tilde{M}_1 \tilde{M}_2 = \tilde{M}_2 \tilde{M}_1 R_q$.

The case (29) for $r = q - q^{-1} = \lambda$ ($R^{(3)} = R_q$) is the quantum Lorentz group of [12, 13] ($L_q^{(1)}$ in the notation of [1]). If $r \neq \lambda$ we obtain a 'gauged' version of it: $R^{(3)} = \mathrm{e}^{\alpha\sigma_2^3} R_q \mathrm{e}^{-\alpha\sigma_2^3}$ ($r = \lambda \mathrm{e}^{2\alpha}$), where the subindex in $\sigma_2^3$ refers to the second space.

The matrix (30) for $t = 1$ and $q \in R$ corresponds to $L_q^{(2)}$ in [1].

The calculations leading to (30)–(33) require assuming $q^2 \neq 1$. However, the solutions for $q \in R$ are also valid in the limit $q = 1$ (see [2]); in this limit ($R^{(1)} = R^{(4)} = I_4$), the case (30) gives the deformed Lorentz group (twisted) of [22]. For $q = -1$, additional solutions appear and, although we shall not discuss these particular cases (see [2]), the associated Minkowski algebras may be obtained as in the general $q$ case.

These results coincide with the classification in [2]: the solutions (30) correspond to equations (13) and (14) in [2]; similarly, (29), (31), (32) and (33) correspond to (74) ($q$ real), (15), (74) ($q$ imaginary) and (16) in [2].

### *(b) Deformed Lorentz groups associated with $SL_h(2)$*

Now let $R^{(1)} = R_h$, equation (10). For $h$ imaginary, $h \in \mathrm{i}R$, the matrix $R_h$ satisfies the reality condition $R_h^* = R_h^{-1}$ $(= \mathcal{P} R_h \mathcal{P})$; this means that $\tilde{M} \equiv M^*$ defines a second copy of $SL_h(2)$ since $R_h M_1^* M_2^* = M_2^* M_1^* R_h$. The value of $h \in C\backslash\{0\}$, however, is not important. Indeed, quantum groups related with two different values of $h \in C$ are equivalent and their $R$ matrices are related by a similarity transformation†; thus, we can take $h \in R$ or even $h = 1$.

Since the entries of $M$ satisfy (11), the 2×2 blocks in $R^{(3)}$ (equation (28)) will now satisfy these commutation relations. This leads to (see [2]) $C = 0$ so that, taking the $h$-'determinant' of $R^{(3)}$ equal $I_2$, the set of commutation relations reduces to

$$AD = I_2 \qquad [A, B] = h(I_2 - A^2)\,. \tag{34}$$

By using them in (28), the following solutions for $R^{(3)}$ are found ($h \in R$):

$$R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad r \in R \tag{35}$$

$$R^{(3)} = \begin{bmatrix} 1 & 0 & -h & 0 \\ -h & 1 & r & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h & 1 \end{bmatrix} \qquad h \in R \quad r \in R\,. \tag{36}$$

*Remarks.* In (36), for $r = h^2$ we have $R^{(3)} = (\mathcal{P} R_h \mathcal{P})^{t_2}$. However, the parameter $r$ can be removed with an appropriate change of basis provided $h \neq 0$. For $h = 0$, this is not possible and constitutes a different case, equation (35). This case is another example where the non-commutativity is solely due to $R^{(3)} \neq I_4$.

The cases (35), (36) correspond to (20) and (21) (cf equation (78) in [2]) in [2].

† Quantum groups associated with $R_h$ and $R_{h=1}$ are related by a similarity transformation defined by the 2×2 matrix $S = \mathrm{diag}(h^{-1/2}, h^{1/2})$: $R_{h=1} = (S \otimes S) R_h (S \otimes S)^{-1}$.

## 4. Minkowski algebras: classification and properties

We now present here, in explicit form, the commutation relations for the generators of the deformed Minkowski spacetimes; they follow easily from (22) using the previous $R^{(3)}$ matrices. We saw in (19) that $R^{(3)\dagger} = R^{(2)} = \mathcal{P}R^{(3)}\mathcal{P}$ and $R^{(4)} = R^{(1)\dagger}$ or $R^{(4)} = (\mathcal{P}R^{(1)\,-1}\mathcal{P})^{\dagger}$ (these two possibilities are the same for $Q = h$). Clearly, equation (22) allows for a factor in one side without impairing its invariance properties. This factor may be selected with the (natural) condition that the resulting Minkowski algebra does not contain generators $\alpha, \beta, \ldots$, with the Grassmann-like property $\alpha^2 = \beta^2 = \cdots = 0$. In terms of $P_{Q+}$, this is tantamount to requiring that $P_{Q+}K_1\hat{R}^{(3)}K_1P^{\dagger}_{Q+}$ must be non-zero. This leads to (cf equation (22)) the equations

$$R_Q K_1 R^{(2)} K_2 = \pm K_2 R^{(3)} K_1 R^{\dagger}_Q \qquad + \text{ for } q, h \in R \quad - \text{ for } q \in \mathrm{i}R\,. \tag{37}$$

In the $q$-case we might also consider $R^{(4)} = (\mathcal{P}R^{(1)\,-1}\mathcal{P})^{\dagger}$. However, using Hecke's condition for $R^{(1)}$ it is seen that this leads to the same algebra as (37) with the restriction $\det_q K = 0$, so that this case may be considered as included in the previous one.

An important ingredient is the centrality of the $Q$-determinant (23), $(\det_Q K)K = K(\det_Q K)$, since it will correspond to the Minkowski length. Using twice (37) we find the following commutation property for three $K$ matrices

$$R_{Q\,13}R_{Q\,23}K_1R^{(2)}_{12}K_2R^{(2)}_{13}R^{(2)}_{23}K_3 = K_3R^{(3)}_{13}R^{(3)}_{23}K_1R^{(2)}_{12}K_2R^{\dagger}_{Q\,13}R^{\dagger}_{Q\,23}\,. \tag{38}$$

Multiplying from the right by $\mathcal{P}_{12}P^{\dagger}_{Q-\,12}$ and by $P_{Q-\,12}$ from the left and using that $R_Q$ and $R^{(3)}$ represent $GL_Q(2)$ and hence have a central $Q$-'determinant' represented by a scalar $2\times2$ matrix we get

$$(\det_Q R_Q)(\det_Q R^{(3)})^{\dagger}\,(\det_Q K)K = (\det_Q R_Q)^{\dagger}(\det_Q R^{(3)})\,K(\det_Q K)\,. \tag{39}$$

The scalar $\det_Q R^{(i)}$ matrices always cancel out in the cases below ($\det_q R_q = qI_2$ and $\det_h R_h = I_2$) assuring the centrality of $\det_Q K$ (as may be checked by direct computation).

### *(a) q-Minkowski spaces associated with $SL_q(2)$*

*(1)* Let us consider the case (29) for $r = \lambda$ (i.e. $R^{(3)} = R_q$, $q$ real). The commutation relations for the entries of $K = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ are

$$\begin{gathered}\alpha\beta = q^{-2}\beta\alpha \qquad [\delta,\beta] = q^{-1}\lambda\alpha\beta \qquad \alpha\gamma = q^2\gamma\alpha\\ [\beta,\gamma] = q^{-1}\lambda(\delta-\alpha)\alpha \qquad [\alpha,\delta] = 0 \qquad [\gamma,\delta] = q^{-1}\lambda\gamma\alpha\end{gathered} \tag{40}$$

they characterize the algebra $\mathcal{M}^{(1)}_q$ ([12–14]; see also [20, 23, 24, 1]). The Minkowski length is given by (23)

$$\det_q K = \alpha\delta - q^2\gamma\beta\,. \tag{41}$$

If $r \neq \lambda$, the commutation relations are slightly different; this, however, corresponds only to an appropriate election of the basis ('gauged' version of this Minkowski space).

*(2)* Let $R^{(3)}$ be given by (30). The centrality of the $q$-determinant implies that $q$ and $t$ are both real or both imaginary. The commutation relation for the entries of $K$ and the $q$-Minkowski length (equation (23)) are (the $+$ sign is for $q, t \in R$ and the $-$ for $q, t \in \mathrm{i}R$)

$$\begin{gathered}q\alpha\beta = \pm t\beta\alpha \qquad t\alpha\gamma = \pm q\gamma\alpha \qquad \alpha\delta = \delta\alpha\\ [\beta,\gamma] = \pm t\lambda\alpha\delta \qquad \beta\delta = \pm qt\delta\beta \qquad \delta\gamma = \pm qt\gamma\delta\end{gathered} \tag{42}$$

$$\det_{q,t} K = \frac{q+q^{-1}}{q \pm q^{-1}}(-q\gamma\beta \pm t\alpha\delta)\,. \tag{43}$$

*Remarks.* For $t = 1$, these commutation relations correspond to the Minkowski algebra $\mathcal{M}_q^{(2)}$ [12, 25, 1] which is isomorphic to the quantum algebra† $GL_q(2)$.

For $q = 1$ and $t$ real, we get the Minkowski space obtained in [22] (denoted $\mathcal{M}^{(3)}$ in [1]). This algebra and the corresponding deformed Poincaré algebra have been shown to be [27] a simple transformation (twisting) of the classical one. As a result, it is possible to remove the non-commuting character of the entries of $K$ [28].

*(3)* Let us take $R^{(3)}$ as in (31) for $r{=}-\lambda$ ($R^{(3)} = \mathcal{P}R_q^{-1}\mathcal{P}$). Then,

$$\begin{aligned} &[\alpha,\beta] = q\lambda\beta\delta \qquad [\alpha,\gamma] = -q\lambda\delta\gamma \qquad [\alpha,\delta] = 0 \\ &[\beta,\gamma] = q\lambda(\alpha-\delta)\delta \qquad \beta\delta = q^2\delta\beta \qquad \gamma\delta = q^{-2}\delta\gamma \end{aligned} \tag{44}$$

$$\det_q K = q^2\alpha\delta - \beta\gamma\,. \tag{45}$$

This algebra may also be identified with the algebra of spacetime derivatives in [14] (see also [23]).

*(4)* Let $R^{(3)}$ be now given by (32). The Minkowski algebra and the central length are given by

$$\begin{aligned} &\alpha\beta = -q^{-2}\beta\alpha \qquad \delta\beta + \beta\delta = r\alpha\beta \qquad \alpha\gamma = -q^2\gamma\alpha \\ &[\beta,\gamma] = -q^{-1}\lambda\delta\alpha + r\alpha^2 \qquad [\alpha,\delta] = 0 \qquad \gamma\delta + \delta\gamma = r\gamma\alpha \end{aligned} \tag{46}$$

$$\det_q K = \frac{-q[2]}{\lambda}(q^{-2}\alpha\delta + \gamma\beta)\,. \tag{47}$$

*(5)* Finally, let $R^{(3)}$ be as in (33). Then

$$\begin{aligned} &\alpha\beta + \beta\alpha = -r\beta\delta \qquad \alpha\gamma + \gamma\alpha = -r\delta\gamma \qquad [\alpha,\delta] = 0 \\ &[\beta,\gamma] = -q\lambda\alpha\delta + r\delta^2 \qquad \beta\delta = -q^2\delta\beta \qquad \gamma\delta = -q^{-2}\delta\gamma \end{aligned} \tag{48}$$

$$\det_q K = \frac{-q[2]}{\lambda}(q^2\alpha\delta + \beta\gamma)\,. \tag{49}$$

*(b) Deformed Minkowski spaces associated with $SL_h(2)$*

*(1)* Let $R^{(3)}$ be given first by (35) and let $R^{(1)} = R_h$, equation (10). Using (37) with the plus sign and (23), we find ($h$ real)

$$\begin{aligned} &[\alpha,\beta] = -h\beta^2 - r\beta\delta + h\delta\alpha - h\beta\gamma + h^2\delta\gamma \qquad [\alpha,\delta] = h(\delta\gamma - \beta\delta) \\ &[\alpha,\gamma] = h\gamma^2 + r\delta\gamma - h\alpha\delta + h\beta\gamma - h^2\beta\delta \qquad [\beta,\delta] = h\delta^2 \\ &[\beta,\gamma] = h\delta(\gamma+\beta) + r\delta^2 \qquad [\gamma,\delta] = -h\delta^2 \end{aligned} \tag{50}$$

$$\det_h K = \frac{2}{h^2+2}(\alpha\delta - \beta\gamma + h\beta\delta)\,. \tag{51}$$

† The Minkowski space of [26] is also a $GL_q(2)$-like space, but is different from the above.

*(2)* Let $R^{(3)}$ be given now by (36) with $r = 0$. In this case

$$[\alpha, \beta] = 2h\alpha\delta + h^2\beta\delta \qquad [\alpha, \delta] = 2h(\delta\gamma - \beta\delta)$$
$$[\alpha, \gamma] = -h^2\delta\gamma - 2h\delta\alpha \qquad [\beta, \delta] = 2h\delta^2 \tag{52}$$
$$[\beta, \gamma] = 3h^2\delta^2 \qquad [\gamma, \delta] = -2h\delta^2$$

$$\det_h K = \frac{2}{h^2+2}(\alpha\delta - \beta\gamma + 2h\beta\delta)\,. \tag{53}$$

*(c) Final remarks*

For all the $Q$-spacetime algebras, time may be defined as proportional to $\mathrm{tr}_Q\, K$ ($= 2x^0$ in the undeformed case). The time generator obtained in this way is central only for $\mathcal{M}_q^{(1)}$ [12–14] and for the Minkowski algebra (44) (in fact, they are isomorphic: the entries of the covariant vector $K^\epsilon$ for $\mathcal{M}_q^{(1)}$ satisfy the commutation relations (44) [1]).

The differential calculus on all the above Minkowski spaces may be easily discussed now along the lines of [1, 23]; one could also investigate the role played in it by the contraction relating [10] the $q$- and $h$-deformations. To conclude, let us mention that the additive braided group structure [19–21] of all these algebras may be easily found. It suffices to impose that equation (37) is also satisfied by the sum $K' + K$ of two copies $K$ and $K'$. Using Hecke's condition ($R_{Q12} = R_{Q21}^{-1} + (\rho - \rho^{-1})\mathcal{P}$) this gives

$$R_Q K_1' R^{(2)} K_2 = \pm K_2 R^{(3)} K_1' (\mathcal{P} R_Q^\dagger \mathcal{P})^{-1} \qquad + \ \text{for } q, h \in R \quad - \ \text{for } q \in \mathrm{i}R \tag{54}$$

which is clearly preserved by (21); for $\mathcal{M}_q^{(1)}$, it reproduces the result of [24].

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